WORKBOOK

Economic Applications 6

Contents

This workbook contains a set of problems whose solutions can be found using mathematical techniques that include calculus applied to optimise a function of two variables and the method of lagrange multipliers. The problems are written using functions that include polynomial (to degree two), rational, algebraic, exponential and logarithmic. Some functions are expressed in terms of parameters or in a general form.

Economic applications include marginal utility, the law of diminishing marginal utility, utility maximisation, the marginal product of labour, the law of diminishing marginal product, the average product of labour, returns to scale, profit maximisation for a multiplant firm and a multiproduct firm, output maximisation, cost minimisation, the demand for inputs, elasticity, the elasticity of substitution, demand and supply, a lump-sum subsidy, the production possibility frontier and opportunity cost.

There are 21 problems together with worked solutions. The answers to them are given separately.

Page border from an edition of ‘The King of the Golden River’ by John Ruskin published as a pamphlet by James Brodie Limited. Illustrator and publication date unknown.

Mathematical Symbols and Letters from the Greek Alphabet used in Workbooks				
Mathematical symbols		***Letters from the Greek Alphabet***		
Symbol	**Meaning**	**Capital**	**Lower case**	**Name**
$\therefore$	therefore	Α	α	alpha
$\approx$	approximately equal to	Β	β	beta
$\equiv$	identically equal to	Δ	δ	delta
$<$	less than	Ε	ε	epsilon
$>$	greater than	Η	η	eta
$\leq$	less than or equal to	Θ	θ	theta
$\geq$	greater than or equal to	Λ	λ	lambda
$\Rightarrow$	implies	Μ	μ	mu
$\pm$	plus or minus	Ν	ν	nu
e	the exponential constant	Π	π	pi
∞	infinity	Ρ	ρ	rho
		$\sum$	σ	sigma

Using this workbook

The purpose of this workbook is to provide a set of problems that involve different aspects of economics and can be solved using the mathematical techniques described on page 1. The problems are arranged randomly in terms of the techniques needed to solve them to help develop an ability to determine the appropriate technique needed to solve a problem. However in general, problems towards the end of the workbook are more difficult to solve and often involve the application of more advanced mathematical techniques.

Worked solutions to the problems are given on pages 9-34. There is almost always more than one way of finding the solution to a problem. In most cases therefore, the worked solution given for a particular problem shows only one possible approach. There is no intention to suggest that the problem *has* to be solved using this approach or that it is necessarily a better approach than the alternatives. However, if there are two or more problems with the same structure (for example two problems that involve solving the same type of equation) the method shown in the worked solution for solving one of them may be different from the method shown for another. This is done to give additional information about solving the problem and to discourage repetition of one method of solution. Answers to all the problems are given on pages 35-36.

The best approach to using this workbook is to try solving some of the early problems and decide if they are appropriate to your level of mathematical ability. If they are too easy try some of the later problems. Solve a problem, check the solution, and then look at the answer given (pages 35-36). If your answer to the problem is not correct, and if *after some more work* you still cannot find the correct answer, look at the worked solution. There is nothing whatsoever to be gained by looking at the worked solution before a serious attempt has been made to try and solve the problem.

Since the problems in this workbook relate to different areas of economics some knowledge of the area of economics relevant to a particular problem is usually needed to solve it.

Problems

1. A firm uses two inputs capital (K) and labour (L) to produce its output. The following table gives the number of units of output that can be obtained from different combinations of inputs used efficiently.

K	L = 16	L = 256	L = 4,096
9	30	60	120
144	120	240	480
2,304	480	960	1,920

(i) Does the information given support constant, increasing or decreasing returns to scale in the production process?

(ii) The production function for this firm takes the form:

$$q = f(L, K) = AL^{\alpha}K^{\beta}$$

Find the values of A, α and β.

(iii) Explain how the returns to scale of the production process for this firm can be determined from the production equation.

2. A firm has a production function given by:

$$q = q(L, K) = 125L^{0.75}K^{0.75}$$

where q = output
L = labour input
K= capital input

(i) What output will the firm produce if it uses 64 units of labour and 4 units of capital?

(ii) By what percentage will output increase if both inputs are increased by 21%?

(iii) What are the returns to scale for this production function?

3. Find the returns to scale for the following production function.

$$q = f(L, K) = (L + K)^{\alpha}(aK + bL)^{1-\alpha}$$

4. A consumer has a utility function of the form:

$$u = u(x, y) = \ln x^{0.3}y^{0.5}$$

where u = index of utility
x = quantity of good X consumed
y = quantity of good Y consumed

(i) Find the marginal utility of each good and evaluate it when $x = 15$ and $y = 8$.

(ii) Is the law of diminishing marginal utility operating for this consumer?

Worked solutions

Note that in a few problems a graph is included in the worked solution given when the problem does not ask for one. This is to help understanding of the economic theory involved.

1. (i) When $K = 9$ and $L = 16$: $q = 30$

When $K = 144$ and $L = 256$: $q = 120$

Both inputs have increased by 16 times their original value $(9 \times 16 = 144)$ and $(16 \times 16 = 256)$. Output has increased by 8 times its original value ($240 = 30 \times 8$)

When $K = 2{,}304$ and $L = 4{,}096$: $q = 1{,}920$

Both inputs have increased by 16 times the previous value $(144 \times 16 = 2{,}304)$ and $(256 \times 16 = 4{,}096)$. Output has increased by 8 times its previous value $(240 \times 8 = 1{,}920)$.

When both inputs increase by 16 times their original values output only increases by 8 times this value, that is, the proportionate increase in output is less than the proportionate increase in inputs so the information given indicates the production process exhibits decreasing returns to scale.

(ii) $q = AL^{\alpha}K^{\beta}$. Since two of the unknowns, α and β are exponents in this equation logarithms can be used to rewrite it so that α and β no longer appear as exponents.

$$\ln q = \ln AL^{\alpha}K^{\beta} = \ln A + \ln L^{\alpha} + \ln K^{\beta} = \ln A + \alpha \ln L + \beta \ln K$$

Using three sets of values from the table gives the equations:

$$\ln 30 = \ln A + \alpha \ln 16 + \beta \ln 9 \qquad (1)$$

$$\ln 60 = \ln A + \alpha \ln 256 + \beta \ln 9 \qquad (2)$$

$$\ln 1{,}920 = \ln A + \alpha \ln 4{,}096 + \beta \ln 2{,}304 \qquad (3)$$

This set of simultaneous equations can be solved as follows.

(2) $-$(1):

$$\ln 60 - \ln 30 = \alpha \ln 256 - \alpha \ln 16$$

$$\ln 2 = \alpha \ln 16 \qquad \text{Using the rule } \ln x - \ln y = \ln\frac{x}{y}$$

$$\alpha = \frac{\ln 2}{\ln 16}$$

$$\alpha = 0.25$$

(3) $-$(2):

$$\ln 1{,}920 - \ln 60 = \alpha \ln 4{,}096 - \ln 256 + \beta \ln 2{,}304 - \beta \ln 9$$

$$\ln 32 = \alpha \ln 16 + \beta \ln 256$$

Substituting for $\alpha = 0.25$ in this equation and then solving for β:

$$\ln 32 = 0.25 \ln 16 + \beta \ln 256$$

$$\ln 32 - 0.25 \ln 16 = \beta \ln 256$$

$$\beta = \frac{\left(\ln \frac{32}{16^{0.25}}\right)}{\ln 256}$$

$$\beta = \frac{\ln 16}{\ln 256}$$

$$\beta = 0.5$$

Substituting $\alpha = 0.25$ and $\beta = 0.5$ in (1):

$$\ln 30 = \ln A + 0.25 \ln 16 + 0.5 \ln 9$$

$$\ln 30 - \ln 16^{0.25} - \ln 9^{0.5} = \ln A$$

$$\ln \frac{30}{16^{0.25} \times 9^{0.5}} = \ln A$$

$$\ln 5 = \ln A$$

$$5 = A$$

The production function is $q = 5L^{0.25}K^{0.5}$.

(iii) The returns to scale are given by the degree of homogeneity of the production function.

$$f(\lambda K, \lambda L) = 5(\lambda L)^{0.25}(\lambda K)^{0.5} = 5\lambda^{0.25}L^{0.25}\lambda^{0.5}K^{0.5} = \lambda^{0.75} f(K, L)$$

This function is homogeneous of degree 0.75. Since the degree of homogeneity is less than one there are decreasing returns to scale.

2. (i) If $L = 64$ and $K = 4$: $q = 125(64)^{0.75}(4)^{0.75} = 8{,}000$

(ii) If inputs increase by 21%: $L = 64(1.21) = 77.44$ and $K = 4(1.21) = 4.84$

$$q = 125(77.44)^{0.75}(4.84)^{0.75} = 10{,}648$$

The percentage change in output is given by:

$$\left(\frac{10{,}648 - 8{,}000}{8{,}000}\right)100 = 33.1\%$$

(iii) Returns to scale are given by the degree of homogeneity of a production function.

$$q(\lambda L, \lambda K) = 125(\lambda L)^{0.75}(\lambda K)^{0.75} = 125\lambda^{0.75}L^{0.75}\lambda^{0.75}K^{0.75} = \lambda^{0.75+0.75}125L^{0.75}K^{0.75}$$

$$= \lambda^{1.5}125L^{0.75}K^{0.75}$$

$$= \lambda^{1.5}q(L, K)$$

This function is homogeneous of degree 1.5. Since this is greater than one this production function exhibits increasing returns to scale.

3. Returns to scale are given by the degree of homogeneity of a production function.

$$f(\lambda L, \lambda K) = (\lambda L + \lambda K)^{\alpha}(a\lambda K + b\lambda L)^{1-\alpha} = \lambda^{\alpha}(L + K)^{\alpha}\lambda^{1-\alpha}(aK + bL)^{1-\alpha}$$

$$= \lambda^{\alpha}\lambda^{1-\alpha}(L + K)^{\alpha}(aK + bL)^{1-\alpha} = \lambda(L + K)^{\alpha}(aK + bL)^{1-\alpha}$$

Using the rule $x^m x^n = x^{m+n}$

$$= \lambda f(L, K)$$

Since λ is raised to the power one this function is homogeneous of degree one and exhibits constant returns to scale.

4. (i) $$\frac{\partial u}{\partial x} = \frac{1}{x^{0.3}y^{0.5}}(0.3x^{-0.7}y^{0.5}) = \frac{0.3}{x} = 0.02 \qquad \text{when} \quad x = 15$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^{0.3}y^{0.5}}(0.5x^{0.3}y^{-0.5}) = \frac{0.5}{y} = 0.0625 \qquad \text{when} \quad y = 8$$

(ii) $$\frac{\partial^2 u}{\partial x^2} = -\frac{0.3}{x^2} < 0$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{0.5}{y^2} < 0$$

The law of diminishing marginal utility is operating for both goods.