(mathematics) $\sqrt{\text{inEconomics}}$

WORKBOOK

Consumer Choice

Contents

The problems in this workbook are based on the simple consumer choice model. The early problems relate to some of the basic concepts of this model and later problems involve solving the complete model.

The mathematical techniques used include differentials, total derivatives and the method of Lagrange multipliers. Some of the functions used are expressed in terms of parameters.

There are 21 problems together with worked solutions. The answers to them are given separately.

Page border from an edition of 'The King of the Golden River' by John Ruskin published as a pamphlet by James Brodie Limited. Illustrator and publication date unknown.

Mathematical symbols and letters from the Greek alphabet used in workbooks				
Mathematical symbols		*Letters from the Greek Alphabet*		
Symbol	**Meaning**	**Capital**	**Lower case**	**Name**
$\therefore$	therefore	A	α	alpha
$\approx$	approximately equal to	B	β	beta
$\equiv$	identically equal to	Δ	δ	delta
$<$	less than	E	ε	epsilon
$>$	greater than	H	η	eta
$\leq$	less than or equal to	Θ	θ	theta
$\geq$	greater than or equal to	Λ	λ	lambda
$\Rightarrow$	implies	M	μ	mu
$\pm$	plus or minus	N	ν	nu
e	the exponential constant	Π	π	pi
∞	infinity	P	ρ	rho
		$\sum$	σ	sigma

Using this workbook

The purpose of this workbook is to provide a set of problems relating to aspects of the basic consumer choice model. To solve the problems in it, therefore, knowledge of this model is needed. Early problems relate to some of the basic concepts of the model and later problems involve solving the complete model and so require the use of more advanced mathematical techniques.

Worked solutions to the problems are given on pages 11-50. There is almost always more than one way of finding the solution to a problem. In most cases, therefore, the worked solution given for a particular problem shows only one possible approach. There is no intention to suggest that the problem *has* to be solved using this approach or that it is necessarily a better approach than alternative approaches. However, if there are two or more problems with the same structure (for example two problems that involve solving the same type of equation) the method shown in the worked solution for solving one of them may be different from the method shown for another. This is done to give additional information about solving the problem and to discourage repetition of one particular method of solution. Answers to all the problems are given on pages 51-56.

Because this workbook contains problems that relate to one economic model and the early problems relate to basic concepts of this model, the best approach to using it is to start at the beginning and work through the problems as they appear. Solve a problem, check the solution, and then look at the answer given (pages 51-56). If your answer to the problem is not correct, and if *after some more work* you still cannot find the correct answer, look at the worked solution. There is nothing whatsoever to be gained by looking at the worked solution before a serious attempt has been made to try and solve the problem.

Problems

1. A consumer spends the available income on one good, *X*, and has a utility function of the following form where *x* is the quantity of *X* consumed and *u* is an index of utility.

   $$u = f(x) = x^{0.75}$$

   Find the marginal utility of *X*. What is marginal utility if 16 units of good *X* are consumed? What is marginal utility if 81 units of good *X* are consumed? Is the law of diminishing marginal utility is operating?

2. A utility function is given by:

   $$u = f(x_1, x_2) = 2x_1^{0.5}x_2^{0.75}$$

   where u = index of utility
   x_i = quantity consumed of good X_i for $i = 1, 2$

   (i) What level of utility will the consumer obtain from consuming each bundle of goods given below?

x_1	1	27	32	36	64
x_2	16	729	4	81	1

   (ii) Will any of the bundles given in (i) lie on the same indifference curve?

   (iii) Which bundle lies on the indifference curve that is farthest from the origin?

   (iv) Draw the indifference curve associated with a level of utility of 16.

   (v) Find a bundle of goods not included in the table in (i) that will yield a level of utility of 16.

   (vi) Find $\frac{dx_2}{dx_1}$ when $u = 16$. Show that $-\frac{dx_2}{dx_1} = \frac{\left(\frac{\partial u}{\partial x_1}\right)}{\left(\frac{\partial u}{\partial x_2}\right)}$

3. A utility function takes the form:

   $$u = 4\ln x_1 x_2$$

   where u = index of utility
   x_i = quantity of good X_i

   (i) Find the marginal utility for each good and determine if the law of diminishing marginal utility is operating.

   (ii) Find the marginal rate of substitution of x_1 for x_2.

Worked solutions

1. $$\frac{du}{dx} = 0.75x^{-0.25}$$

When $x = 16$: $\dfrac{du}{dx} = 0.75x^{-0.25} = \dfrac{0.75}{(16)^{0.25}} = 0.375$

When $x = 81$: $\dfrac{du}{dx} = 0.75x^{-0.25} = \dfrac{0.75}{(81)^{0.25}} = 0.25$

$$\frac{d\left(\frac{du}{dx}\right)}{dx} = \frac{d^2u}{dx^2} = (-0.25)0.75x^{-1.25} = -0.1875x^{-1.25} < 0 \quad \text{for} \quad x > 0$$

Since $\dfrac{d^2u}{dx^2} < 0$ for $x > 0$ the law of diminishing marginal utility is operating.

2. (i)

x_1	1	27	32	36	64
x_2	16	729	4	81	1
$u = 2x_1^{0.5}x_2^{0.75}$	16	1,458	32	324	16

(ii) Let a bundle be represented by (x_1, x_2). The bundles $(1,16)$ and $(64,1)$ yield the same level of utility (16) and therefore lie on the same indifference curve.

(iii) The bundle $(27, 729)$ yields the highest level of utility and so lies on the indifference curve that is further from the origin than those associated with the other four bundles.

(iv) $$u = 2x_1^{0.5}x_2^{0.75}$$

When $u = 16$:

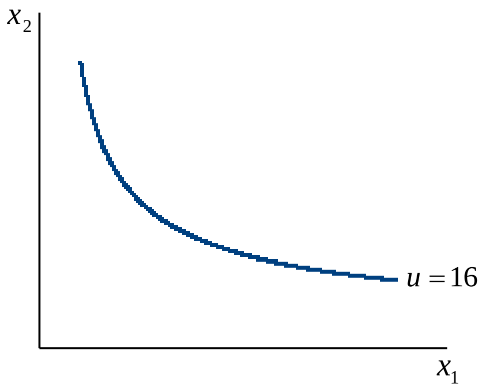
$$16 = 2x_1^{0.5}x_2^{0.75}$$

To draw the graph of the relationship between x_1 and x_2, rearrange this implicit equation to express one variable in terms of the other.

$$\frac{16}{2x_1^{0.5}} = x_2^{0.75}$$

$$(x_2^{0.75})^{\frac{1}{0.75}} = \left(\frac{8}{x_1^{0.5}}\right)^{\frac{1}{0.75}}$$

$$x_2 = \frac{16}{x_1^{\frac{2}{3}}}$$



(v) Any pair of values (x_1, x_2) that satisfies the equation of the indifference curve, $x_2 = \dfrac{16}{x_1^{\frac{2}{3}}}$, will yield a level of utility of 16, for example $(8,4)$.

(vi) $$\frac{dx_2}{dx_1} = -\frac{2}{3}\left(\frac{16}{x_1^{\frac{5}{3}}}\right) = -\frac{32}{3x_1^{\frac{5}{3}}} \quad \Rightarrow \quad -\frac{dx_2}{dx_1} = \frac{32}{3x_1^{\frac{5}{3}}}$$

$$\frac{\partial u}{\partial x_1} = (0.5)2x_1^{-0.5}x_2^{0.75} = x_1^{-0.5}x_2^{0.75}$$

$$\frac{\partial u}{\partial x_2} = (0.75)2x_1^{0.5}x_2^{-0.25} = 1.5x_1^{0.5}x_2^{-0.25}$$

$$\frac{\left(\dfrac{\partial u}{\partial x_1}\right)}{\left(\dfrac{\partial u}{\partial x_2}\right)} = \frac{x_1^{-0.5}x_2^{0.75}}{1.5x_1^{0.5}x_2^{-0.25}} = \frac{2x_2}{3x_1}$$

To express this ratio in terms of x_1, substitute for x_2 in terms of x_1 from the equation of the indifference curve.

$$\frac{\left(\dfrac{\partial u}{\partial x_1}\right)}{\left(\dfrac{\partial u}{\partial x_2}\right)} = \frac{2x_2}{3x_1} = \frac{2\left(\dfrac{16}{x_1^{\frac{2}{3}}}\right)}{3x_1} = \frac{32}{3x_1^{\frac{5}{3}}}$$

3. (i) $$u = 4\ln x_1x_2 = \ln x_1^4x_2^4 = \ln x_1^4 + \ln x_2^4 = 4\ln x_1 + 4\ln x_2$$

Using the rules $\log_b x^a = a\log_b x$ and $\log_b xy = \log_b x + \log_b y$.

$$\frac{\partial u}{\partial x_1} = \frac{4}{x_1} \qquad \frac{\partial^2 u}{\partial x_1^2} = -\frac{4}{x_1^2} < 0$$

$$\frac{\partial u}{\partial x_2} = \frac{4}{x_2} \qquad \frac{\partial^2 u}{\partial x_2^2} = -\frac{4}{x_2^2} < 0$$

The law of diminishing marginal utility is operating for both goods.

(ii) The marginal rate of substitution of x_1 for x_2 (MRS_{x_1,x_2}) is given by:

$$MRS_{x_1,x_2} = -\frac{dx_2}{dx_1}$$

It is defined along an indifference curve, that is, it describes an aspect of the relationship between x_1 and x_2 when $u = \bar{u}$. The marginal rate of substitution can be found using differentials or by finding the equation of an indifference curve.

The total differential of the utility function is given by:

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = \frac{4}{x_1} dx_1 + \frac{4}{x_2} dx_2$$

Since utility is constant along an indifference curve $du = 0$:

$$0 = \frac{4}{x_1} dx_1 + \frac{4}{x_2} dx_2$$

$$-\frac{4}{x_2} dx_2 = \frac{4}{x_1} dx_1$$

$$-\frac{dx_2}{dx_1} = \frac{\left(\frac{4}{x_1}\right)}{\left(\frac{4}{x_2}\right)}$$

$$-\frac{dx_2}{dx_1} = \frac{x_2}{x_1}$$

Alternatively find the equation that defines an indifference curve. Along an indifference curve $u = \bar{u}$:

$$\bar{u} = 4\ln x_1 + 4\ln x_2$$

Rearrange this equation to express x_2 in terms of x_1.

$$4\ln x_2 = \bar{u} - 4\ln x_1$$

$$\ln x_2 = 0.25\bar{u} - \ln x_1$$

Taking the antilogarithm of both sides:

$$x_2 = e^{0.25\bar{u} - \ln x_1}$$

$$\frac{dx_2}{dx_1} = -\frac{1}{x_1} e^{0.25\bar{u} - \ln x_1} \quad \Rightarrow \quad -\frac{dx_2}{dx_1} = \frac{1}{x_1} e^{0.25\bar{u} - \ln x_1}$$ Using the chain rule

To express this derivative in terms of x_1 and x_2 substitute $4\ln x_1 + 4\ln x_2$ for $\bar{u}$:

$$-\frac{dx_2}{dx_1} = \frac{1}{x_1} e^{0.25\bar{u} - \ln x_1} = \frac{1}{x_1} e^{0.25(4\ln x_1 + 4\ln x_2) - \ln x_1} = \frac{1}{x_1} e^{\ln x_2} = \frac{x_2}{x_1}$$ Since $b^{\log_b x} = x$