(mathematics) $\sqrt{\text{inEconomics}}$

Mathematical Methods in Economics: Problems and Solutions

Chapter 17

Worked Solutions to Problems on

Constrained Optimisation II

8.1 Maximise $U = u(x, y) = x^{\alpha} y^{\beta}$

Subject to $p_x x + p_y y = m$

$$\ell(x, y, \lambda) = x^{\alpha} y^{\beta} - \lambda(p_x x + p_y y - m)$$

First-order conditions for a stationary point:

$$\frac{\partial \ell}{\partial x} = \alpha x^{\alpha-1} y^{\beta} - \lambda p_x = 0 \qquad (1)$$

$$\frac{\partial \ell}{\partial y} = \beta x^{\alpha} y^{\beta-1} - \lambda p_y = 0 \qquad (2)$$

$$\frac{\partial \ell}{\partial \lambda} = -(p_x x + p_y y - m) = 0 \qquad (3)$$

One way to solve this set of equations is as follows[1].

(1)×x $\quad \alpha x^{\alpha-1} y^{\beta} x - \lambda p_x x = 0 \quad \Rightarrow \quad x = \dfrac{\alpha x^{\alpha} y^{\beta}}{\lambda p_x} = \dfrac{\alpha U}{\lambda p_x}$ Substituting U for $x^{\alpha} y^{\beta}$

(2)×y $\quad \beta x^{\alpha} y^{\beta-1} y - \lambda p_y y = 0 \quad \Rightarrow \quad y = \dfrac{\beta x^{\alpha} y^{\beta}}{\lambda p_y} = \dfrac{\beta U}{\lambda p_y}$

Substituting for x and y in (3):

$$p_x\left(\frac{\alpha U}{\lambda p_x}\right) + p_y\left(\frac{\beta U}{\lambda p_y}\right) = m \quad \Rightarrow \quad \lambda = \frac{(\alpha+\beta)U}{m}$$

$$\therefore \quad x = \frac{\alpha U}{\left(\frac{(\alpha+\beta)U}{m}\right)p_x} = \frac{\alpha m}{(\alpha+\beta)p_x} \quad \text{and} \quad y = \frac{\beta U}{\left(\frac{(\alpha+\beta)U}{m}\right)p_y} = \frac{\beta m}{(\alpha+\beta)p_y}$$

Since $\alpha + \beta = 1$: $\quad x = \dfrac{\alpha m}{p_x}$ and $y = \dfrac{\beta m}{p_y}$

Second-order conditions:

$$\begin{vmatrix} 0 & -p_x & -p_y \\ -p_x & \alpha(\alpha-1)x^{\alpha-2}y^{\beta} & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ -p_y & \alpha\beta x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)x^{\alpha}y^{\beta-2} \end{vmatrix}$$

$$= 0\begin{vmatrix} \alpha(\alpha-1)x^{\alpha-2}y^{\beta} & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ \alpha\beta x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)x^{\alpha}y^{\beta-2} \end{vmatrix} - (-p_x)\begin{vmatrix} -p_x & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ -p_y & \beta(\beta-1)x^{\alpha}y^{\beta-2} \end{vmatrix}$$

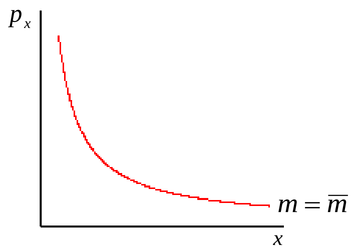
$$+ (-p_y)\begin{vmatrix} -p_x & \alpha(\alpha-1)x^{\alpha-2}y^{\beta} \\ -p_y & \alpha\beta x^{\alpha-1}y^{\beta-1} \end{vmatrix}$$

$$= p_x((-p_x)(\beta(\beta-1)x^{\alpha}y^{\beta-2}) - (-p_y)(\alpha\beta x^{\alpha-1}y^{\beta-1}))$$

$$- p_y((-p_x)(\alpha\beta x^{\alpha-1}y^{\beta-1}) - (-p_y)(\alpha(\alpha-1)x^{\alpha-2}y^{\beta}))$$

$$= 2p_x p_y \alpha\beta x^{\alpha-1}y^{\beta-1} - p_x^2\beta(\beta-1)x^{\alpha}y^{\beta-2} - p_y^2\alpha(\alpha-1)x^{\alpha-2}y^{\beta}$$

For $0 < \alpha, \beta < 1$ and positive values of the variables this expression is positive so utility is maximised when $x = \frac{\alpha m}{p_x}$ and $y = \frac{\beta m}{p_y}$. The demand curve for good X takes the form:



8.2 (i) Let R = total revenue. $\pi = R - C$

$$R = p_1 q_1 + p_2 q_2 = (20 - \tfrac{1}{4}q_1)q_1 + (17 - \tfrac{1}{2}q_2)q_2 = 20q_1 - \tfrac{1}{4}q_1^2 + 17q_2 - \tfrac{1}{2}q_2^2$$

$$\pi = 20q_1 - \tfrac{1}{4}q_1^2 + 17q_2 - \tfrac{1}{2}q_2^2 - (115 + 2q)$$

$$= 20q_1 - \tfrac{1}{4}q_1^2 + 17q_2 - \tfrac{1}{2}q_2^2 - (115 + 2(q_1 + q_2))$$

$$= 18q_1 - \tfrac{1}{4}q_1^2 + 15q_2 - \tfrac{1}{2}q_2^2 - 115$$

First-order conditions for a stationary point:

$$\frac{\partial \pi}{\partial q_1} = 18 - \tfrac{1}{2}q_1 = 0 \qquad (1) \qquad \Rightarrow \qquad q_1 = 36$$

$$\frac{\partial \pi}{\partial q_2} = 15 - q_2 = 0 \qquad (2) \qquad \Rightarrow \qquad q_2 = 15$$

Second-order conditions:

$$\frac{\partial^2 \pi}{\partial q_1^2} = -\frac{1}{2} < 0 \qquad \frac{\partial^2 \pi}{\partial q_1 \partial q_2} = 0$$

$$\frac{\partial^2 \pi}{\partial q_2^2} = -1 < 0$$

$$\left(\frac{\partial^2 \pi}{\partial q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial q_2^2}\right) - \left(\frac{\partial^2 \pi}{\partial q_1 \partial q_2}\right)^2 = (-\tfrac{1}{2})(-1) - (0)^2 = \tfrac{1}{2} > 0$$

Profit is maximised when $q_1 = 36$ and $q_2 = 15$.

When $q_1 = 36$ and $q_2 = 15$: $\pi = 18(36) - \tfrac{1}{4}(36)^2 + 15(15) - \tfrac{1}{2}(15)^2 - 115 = 321.5$

(ii) The problem now is to maximise the profit function subject to the constraint $p_1 = p_2 + 3$. However since the objective function is expressed in terms of q_1 and q_2, it is necessary to express the constraint in terms of q_1 and q_2:

$$p_1 = p_2 + 3 \qquad \Rightarrow \qquad 20 - \tfrac{1}{4}q_1 = 17 - \tfrac{1}{2}q_2 + 3 \qquad \Rightarrow \qquad \tfrac{1}{2}q_2 - \tfrac{1}{4}q_1 = 0$$

The problem may therefore be expressed as:

Maximise $\quad \pi = 18q_1 - \tfrac{1}{4}q_1^2 + 15q_2 - \tfrac{1}{2}q_2^2 - 115$

Subject to $\qquad \frac{1}{2}q_2 - \frac{1}{4}q_1 = 0$

$$\ell(q_1, q_2, \lambda) = 18q_1 - \frac{1}{4}q_1^2 + 15q_2 - \frac{1}{2}q_2^2 - 115 + \lambda(\frac{1}{2}q_2 - \frac{1}{4}q_1)$$

First-order conditions for a stationary point:

$$\frac{\partial \ell}{\partial q_1} = 18 - \frac{1}{2}q_1 - \frac{1}{4}\lambda = 0 \qquad (1) \qquad \Rightarrow \qquad q_1 = 36 - \frac{1}{2}\lambda$$

$$\frac{\partial \ell}{\partial q_2} = 15 - q_2 + \frac{\lambda}{2} = 0 \qquad (2) \qquad \Rightarrow \qquad q_2 = 15 + \frac{\lambda}{2}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{1}{2}q_2 - \frac{1}{4}q_1 = 0 \qquad (3)$$

To solve this set of equations, substitute for q_1 and q_2 in (3):

$$\frac{1}{2}[15 + \frac{\lambda}{2}] - \frac{1}{4}[36 - \frac{1}{2}\lambda] = 0 \qquad \Rightarrow \qquad \lambda = 4$$

$$\therefore \qquad q_1 = 36 - \frac{1}{2}(4) = 34 \qquad \text{and} \qquad q_2 = 15 + \frac{4}{2} = 17$$

Second-order conditions:

$$\begin{vmatrix} 0 & -\frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -1 \end{vmatrix} = 0\begin{vmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{vmatrix} - (-\frac{1}{4})\begin{vmatrix} -\frac{1}{4} & 0 \\ \frac{1}{2} & -1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix}$$

$$= \frac{1}{4}(-\frac{1}{4})(-1) + \frac{1}{2}(-(\frac{1}{2})(-\frac{1}{2}) = \frac{3}{16} > 0$$

Profit is maximised when $q_1 = 34$ and $q_2 = 17$.

When $q_1 = 34$ and $q_2 = 17$:

$$\pi = 18(34) - \frac{1}{4}(34)^2 + 15(17) - \frac{1}{2}(17)^2 - 115 = 318.5$$

The change in profit is: $318.5 - 321.5 = -3$.