(*mathematics*) √ *inEconomics*

Mathematical Methods in Economics: Problems and Solutions

Chapter 16

Worked Solutions to Problems on Constrained Optimisation I

7.1 Minimise[1] $C = 6x^2 + 3y^2$ (objective function)

Subject to $x + y = 21$ (constraint)

$$\ell(x, y, \lambda) = 6x^2 + 3y^2 + \lambda(21 - x - y)$$

First-order conditions for a stationary point: [2]

$$\frac{\partial \ell}{\partial x} = 12x - \lambda = 0 \quad (1)$$

$$\frac{\partial \ell}{\partial y} = 6y - \lambda = 0 \quad (2)$$

$$\frac{\partial \ell}{\partial \lambda} = 21 - x - y = 0 \quad (3)$$

One way to solve this set of equations for x, y and λ is as follows:

From (1): $x = \frac{\lambda}{12}$ From (2): $y = \frac{\lambda}{6}$

Substituting for x and y in (3) reduces the system to an equation in λ only:

$$\frac{\lambda}{12} + \frac{\lambda}{6} = 21 \quad \Rightarrow \quad \lambda = 84$$

$$\therefore \quad x = \frac{84}{12} = 7 \quad \text{and} \quad y = \frac{84}{6} = 14$$

Second-order conditions: [3]

$$\begin{vmatrix} 0 & -1 & -1 \\ -1 & 12 & 0 \\ -1 & 0 & 6 \end{vmatrix} = 0\begin{vmatrix} 12 & 0 \\ 0 & 6 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ -1 & 6 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 12 \\ -1 & 0 \end{vmatrix}$$

$$= 0(12(6) - (0)^2) + 1((-1)6 - (-1)0) - 1((-1)0 - (-1)12) = -18 < 0$$

The firm will minimise cost by producing 7 units of good X and 14 units of good Y. Minimum cost is $C = 6(7)^2 + 3(14)^2 = 882$.

7.2 Maximise $U = xy$

Subject to $5x + 2y = 20$

$$\ell(x, y, \lambda) = xy - \lambda(5x + 2y - 20)$$

First-order conditions for a stationary point:

$$\frac{\partial \ell}{\partial x} = y - 5\lambda = 0 \quad (1) \quad \Rightarrow \quad y = 5\lambda$$

$$\frac{\partial \ell}{\partial y} = x - 2\lambda = 0 \quad (2) \quad \Rightarrow \quad x = 2\lambda$$

$$\frac{\partial \ell}{\partial \lambda} = -(5x + 2y - 20) = 0 \qquad (3)$$

Substituting for *x* and *y* in (3):

$$5(2\lambda) + 2(5\lambda) = 20 \qquad \Rightarrow \qquad \lambda = 1$$

$\therefore \qquad x = 2(1) = 2 \qquad \text{and} \qquad y = 5(1) = 5.$

Second-order conditions:

$$\begin{vmatrix} 0 & -5 & -2 \\ -5 & 0 & 1 \\ -2 & 1 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - (-5)\begin{vmatrix} -5 & 1 \\ -2 & 0 \end{vmatrix} + (-2)\begin{vmatrix} -5 & 0 \\ -2 & 1 \end{vmatrix}$$

$$= 5((-5)0 - (-2)1) - 2((-5)1 - (-2)0) = 20 > 0$$

Utility is maximised when $x = 2$ and $y = 5$. The marginal utility of money is given by $\lambda = 1$.