(mathematics) √ inEconomics

Mathematical Methods in Economics: Problems and Solutions

Chapter 15

Worked Solutions to Problems on Optimising a Function of Two Variables

6.1 Since the objective of the monopolist is to determine the quantity that should be sold in each market in order to maximise profit, it is necessary to express the profit function in terms of q_1 and q_2 and then find for what values of q_1 and q_2 the profit function is maximised. Let total revenue be denoted by *R*, total revenue from market *i* by R_i for $i = 1,2$ and total cost by *C*.

$$\pi = R - C = R_1 + R_2 - C$$

Total revenue from market 1: $R_1 = p_1 q_1 = (100 - q_1)q_1 = 100q_1 - q_1^2$

Total revenue from market 2: $R_2 = p_2 q_2 = (84 - q_2)q_2 = 84q_2 - q_2^2$

$$\pi = 100q_1 - q_1^2 + 84q_2 - q_2^2 - (600 + 4q_1 + 4q_2) = 96q_1 - q_1^2 + 80q_2 - q_2^2 - 600$$

First-order conditions[1] for a stationary point:

$$\frac{\partial \pi}{\partial q_1} = 96 - 2q_1 = 0 \qquad \Rightarrow \qquad q_1 = 48$$

$$\frac{\partial \pi}{\partial q_2} = 80 - 2q_2 = 0 \qquad \Rightarrow \qquad q_2 = 40$$

Second-order conditions:

$$\frac{\partial^2 \pi}{\partial q_1^2} = -2 < 0 \qquad \frac{\partial^2 \pi}{\partial q_1 \partial q_2} = 0$$

$$\frac{\partial^2 \pi}{\partial q_2^2} = -2 < 0$$

$$\left(\frac{\partial^2 \pi}{\partial q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial q_2^2}\right) - \left(\frac{\partial^2 \pi}{\partial q_1 \partial q_2}\right)^2 = (-2)(-2) - (0)^2 = 4 > 0$$

The second-order conditions for a maximum[2] are fulfilled so the monopolist will maximise profit by selling 48 units in market 1 and 40 units in market 2.

6.2 (i) Expressing the profit function in terms of q_1 and q_2:

$$\pi = R_1 + R_2 - C$$

$$= p_1 q_1 + p_2 q_2 - C$$

$$= (180 - 2q_1)q_1 + (132 - 3q_2)q_2 - [2(q_1 + q_2)^2 + 20(q_1 + q_2) + 85]$$

$$= 160q_1 - 4q_1^2 + 112q_2 - 5q_2^2 - 4q_1 q_2 - 85$$

First-order conditions for a stationary point:

$$\frac{\partial \pi}{\partial q_1} = 160 - 8q_1 - 4q_2 = 0 \qquad (1)$$

$$\frac{\partial \pi}{\partial q_2} = 112 - 10q_2 - 4q_1 = 0 \qquad (2)$$

Solving this set of linear simultaneous equations gives $q_1 = 18$ and $q_2 = 4$.

Second-order conditions:

$$\frac{\partial^2 \pi}{\partial q_1^2} = -8 < 0 \qquad \frac{\partial^2 \pi}{\partial q_1 \partial q_2} = -4$$

$$\frac{\partial^2 \pi}{\partial q_2^2} = -10 < 0$$

$$\left(\frac{\partial^2 \pi}{\partial q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial q_2^2}\right) - \left(\frac{\partial^2 \pi}{\partial q_1 \partial q_2}\right)^2 = (-8)(-10) - (-4)^2 = 64 > 0$$

Profit is maximised when $q_1 = 18$ and $q_2 = 4$.

When $q_1 = 18$: $p_1 = 180 - 2(18) = 144$

$q_2 = 4$: $p_2 = 132 - 3(4) = 120$

(ii) Let the inverse demand (average revenue) function in market 1 be given by $p_1 = a - bq_1$.

Total revenue in market 1: $R_1 = (a - bq_1)q_1 = aq_1 - bq_1^2$

Marginal revenue in market1: $\frac{dR_1}{dq_1} = a - 2bq_1$

When $q_1 = 13$, $p_1 = 115.5$ and marginal revenue in the market is 96. Substituting these values into the equations that define the inverse demand function and marginal revenue function gives two equations from which to solve for *a* and *b*:

$115.5 = a - b13$ (1)

$96 = a - b26$ (2)

Solving this set of equations gives *a* =135 and *b* = 1.5. After preferences change the inverse demand function in market 1 is given by:

$$p_1 = 135 - 1.5q_1$$

The demand function is:

$$q_1 = 90 - \tfrac{2}{3} p_1$$

6.3 (i)[3] $q_1 = 120 - 0.25q_2$

$q_2 = 192 - 0.4q_1$

The equilibrium output of each firm in this market is given by the values of q_1 and q_2 that satisfy these equations simultaneously. Solving these equations gives $q_1 = 80$ and $q_2 = 160$.

Total quantity supplied to the market: $q_1 + q_2 = 80 + 160 = 240$

Market price can be found from the inverse demand function:

$$q = 500 - 10p \quad \Rightarrow \quad p = 50 - 0.1q .$$

When q = 240: $p = 50 - 0.1(240) = 26$

Profit to firm 1:

$$\pi_1 = pq_1 - (0.1q_1^2 + 2q_1) = 26(80) - (0.1(80)^2 + 2(80)) = 1{,}280$$

Profit to firm 2:

$$\pi_2 = pq_2 - (0.025q_2^2 + 2q_2) = 26(160) - (0.025(160)^2 + 2(160)) = 3{,}200$$

(ii) The joint profit function is given by:

$$\pi = p(q_1 + q_2) - (C_1 + C_2)$$

$$= [50 - 0.1(q_1 + q_2)](q_1 + q_2) - (0.1q_1^2 + 2q_1 + 0.025q_2^2 + 2q_2)$$

$$= 48q_1 + 48q_2 - 0.2q_1^2 - 0.125q_2^2 - 0.2q_1q_2$$

First-order conditions for a stationary point:

$$\frac{\partial \pi}{\partial q_1} = 48 - 0.4q_1 - 0.2q_2 = 0$$

$$\frac{\partial \pi}{\partial q_2} = 48 - 0.2q_1 - 0.25q_2 = 0$$

Solving this set of simultaneous equations yields: $q_1 = 40$ and $q_2 = 160$.

Second-order conditions:

$$\frac{\partial^2 \pi}{\partial q_1^2} = -0.4 < 0 \qquad \frac{\partial^2 \pi}{\partial q_1 \partial q_2} = -0.2$$

$$\frac{\partial^2 \pi}{\partial q_2^2} = -0.25 < 0$$

$$\left(\frac{\partial^2 \pi}{\partial q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial q_2^2}\right) - \left(\frac{\partial^2 \pi}{\partial q_1 \partial q_2}\right)^2 = (-0.4)(-0.25) - (-0.2)^2 = 0.06 > 0$$

Joint profit is maximised[4] when $q_1 = 40$ and $q_2 = 160$.

Quantity supplied to the market:

$$q_1 + q_2 = 40 + 160 = 200$$

When q = 200: $p = 50 - 0.1(200) = 30$

Profit to firm 1:

$$\pi_1 = pq_1 - (0.1q_1^2 + 2q_1) = 30(40) - (0.1(40)^2 + 2(40)) = 960$$

Profit to firm 2:

$$\pi_2 = pq_2 - (0.025q_2^2 + 2q_2) = 30(160) - (0.025(160)^2 + 2(160)) = 3{,}840$$

To maintain the cartel[5] firm 2 must pay firm1 at least $1{,}280 - 960 = 320$.