(mathematics) √ in Economics

Mathematical Methods in Economics: Problems and Solutions

Chapter 14

Worked Solutions to Problems on Partial Differentiation

5.1 Marginal cost of product 1: $\dfrac{\partial C}{\partial q_1} = 6q_1^2 + 4.8q_1q_2$

Marginal cost of product 2: $\dfrac{\partial C}{\partial q_2} = 2q_2^{-0.5} + 2.4q_1^2$

5.2 (i) The marginal product of labour is given by: $\dfrac{\partial q}{\partial L} = \left(\dfrac{1}{2}\right)12L^{\frac{1}{2}-1}K^{\frac{1}{4}} = 6L^{-\frac{1}{2}}K^{\frac{1}{4}} = \dfrac{6K^{\frac{1}{4}}}{L^{\frac{1}{2}}}$

The marginal product of capital is given by: $\dfrac{\partial q}{\partial K} = \left(\dfrac{1}{4}\right)12L^{\frac{1}{2}}K^{\frac{1}{4}-1} = 3L^{\frac{1}{2}}K^{-\frac{3}{4}} = \dfrac{3L^{\frac{1}{2}}}{K^{\frac{3}{4}}}$

(ii) $$\frac{\partial\left(\dfrac{\partial q}{\partial L}\right)}{\partial L} = \frac{\partial^2 q}{\partial L^2} = \left(-\frac{1}{2}\right)6L^{-\frac{1}{2}-1}K^{\frac{1}{4}} = -3L^{-\frac{3}{2}}K^{\frac{1}{4}} = -\frac{3K^{\frac{1}{4}}}{L^{\frac{3}{2}}} < 0 \quad \text{for } L, K > 0$$

The marginal product of labour falls as L increases so the law of diminishing marginal product is operating for labour.

$$\frac{\partial\left(\dfrac{\partial q}{\partial K}\right)}{\partial K} = \frac{\partial^2 q}{\partial K^2} = \left(-\frac{3}{4}\right)3L^{\frac{1}{2}}K^{-\frac{3}{4}-1} = -\frac{9}{4}L^{-\frac{3}{2}}K^{-\frac{7}{4}} = -\frac{9L^{\frac{1}{2}}}{4K^{\frac{7}{4}}} < 0 \quad \text{for } L, K > 0$$

The marginal product of capital falls as K increases so the law of diminishing marginal product is operating for capital.

5.3 (i)[1] $u = u(q_1, q_2) = \frac{3}{4}\ln q_1 + \frac{1}{3}\ln q_2$

The marginal utility of good 1 is given by[2]: $\dfrac{\partial u}{\partial q_1} = \dfrac{3}{4q_1}$

The marginal utility of good 2 is given by[3]: $\dfrac{\partial u}{\partial q_2} = \dfrac{1}{3q_2}$

(ii) The law of diminishing marginal utility is operating if the marginal utility of a good falls as the quantity consumed increases.

$$\frac{\partial\left(\dfrac{\partial u}{\partial q_1}\right)}{\partial q_1} = \frac{\partial^2 u}{\partial q_1^2} = -\frac{3}{4q_1^2} < 0 \qquad \forall \qquad q_1 \neq 0$$

Since the change in the marginal utility is negative, it is falling as the quantity of the good is increased and so the law of diminishing marginal utility is operating.

$$\frac{\partial\left(\dfrac{\partial u}{\partial q_2}\right)}{\partial q_2} = \frac{\partial^2 u}{\partial q_2^2} = -\frac{1}{3q_2^2} < 0 \qquad \forall \qquad q_2 \neq 0$$

The law of diminishing marginal utility is also operating for good 2.

(iii) The conditions required for the law of diminishing marginal utility to be operating for the utility function $U = f(q_1, q_2)$ are:

$$f_{q_1q_1} < 0 \quad \text{and} \quad f_{q_2q_2} < 0$$

5.4 (i) Marginal cost for good 1: $\dfrac{\partial C}{\partial q_1} = 1.2q_1^2 - 12q_1 + 32 + 4q_2$

Marginal cost for good 2: $\dfrac{\partial C}{\partial q_2} = 1.08q_2^2 - 8.64q_2 + 20 + 4q_1$

(ii)

$$\frac{\partial\left(\dfrac{\partial C}{\partial q_1}\right)}{\partial q_1} = \frac{\partial^2 q_1}{\partial q_1^2} = 2.4q_1 - 12 \begin{cases} < 0 & \text{marginal cost is decreasing} \\ > 0 & \text{marginal cost is increasing} \end{cases}$$

$$2.4q_1 - 12 < 0 \quad \Rightarrow \quad q_1 < 5$$

$$2.4q_1 - 12 > 0 \quad \Rightarrow \quad q_1 > 5$$

For good 1 marginal cost is decreasing for $0 \le q_1 < 5$ and increasing for $q_1 > 5$.

$$\frac{\partial\left(\dfrac{\partial C}{\partial q_2}\right)}{\partial q_2} = \frac{\partial^2 q_2}{\partial q_2^2} = 2.16q_2 - 8.64 \begin{cases} < 0 & \text{marginal cost is decreasing} \\ > 0 & \text{marginal cost is increasing} \end{cases}$$

$$2.16q_2 - 8.64 < 0 \quad \Rightarrow \quad q_2 < 4$$

$$2.16q_2 - 8.64 > 0 \quad \Rightarrow \quad q_2 > 4$$

For good 2 marginal cost is decreasing for $0 \le q_2 < 4$ and increasing for $q_2 > 4$.

5.5 (i)

$$\frac{\partial C}{\partial q} = (\tfrac{4}{3})10w^{\frac{1}{4}}r^{\frac{1}{2}}q^{\frac{4}{3}-1} = \tfrac{40}{3}w^{\frac{1}{4}}r^{\frac{1}{2}}q^{\frac{1}{3}}$$

(ii)

$$\frac{\partial^2 C}{\partial q^2} = (\tfrac{1}{3})(\tfrac{4}{3})10w^{\frac{1}{4}}r^{\frac{1}{2}}q^{\frac{1}{3}-1} = \tfrac{40}{9}w^{\frac{1}{4}}r^{\frac{1}{2}}q^{-\frac{2}{3}} > 0 \quad \text{for } q > 0$$

The marginal cost function has a positive slope at all levels of output so it is monotonically increasing.

(iii) The elasticity of C with respect to w is given by:

$$\frac{\partial C}{\partial w} \cdot \frac{w}{C} = (\tfrac{1}{4})10w^{-\frac{3}{4}}r^{\frac{1}{2}}q^{\frac{4}{3}}\left(\frac{w}{10w^{\frac{1}{4}}r^{\frac{1}{2}}q^{\frac{4}{3}}}\right) = \frac{1}{4}$$

5.6 The own-price elasticity of demand is given by:

$$\eta = \frac{\partial q_a^d}{\partial p_a} \cdot \frac{p_a}{q_a^d} = -\alpha_1\left(\frac{p_a}{\alpha_0 - \alpha_1 p_a + \alpha_2 p_b + \alpha_3 Y}\right)$$

The cross-price elasticity of demand is given by:

$$\frac{\partial q_a^d}{\partial p_b} \cdot \frac{p_b}{q_a^d} = \alpha_2 \left(\frac{p_b}{\alpha_0 - \alpha_1 p_a + \alpha_2 p_b + \alpha_3 Y} \right)$$

The income elasticity of demand is given by:

$$\frac{\partial q_a^d}{\partial Y} \cdot \frac{Y}{q_a^d} = \alpha_3 \left(\frac{Y}{\alpha_0 - \alpha_1 p_a + \alpha_2 p_b + \alpha_3 Y} \right)$$

The own-price elasticity of demand is negative indicating that good *A* has a downward sloping demand function. The cross-price elasticity of demand is positive so goods *A* and *B* are substitutes and as the income elasticity is also positive, good *A* is a normal good.